VARIATION OF PARAMETERS.

Euler devised method known as the variation of parameters to solve non-homogeneous DE.

$$Ly = g(x)$$

which is a linear, 2^{nd} order, non-homogeneous DE w̄ constant coefficients.

$$a_2 y'' + a_1 y' + a_0 y = g(x)$$

if we divide by a_2, it reduces to standard form.

$$y'' + py' + qy = f(x)$$

where $p, q, f(x)$, are continuous functions of x.

$\therefore$

$$Ly(x) = (D^2 + pD + q)y(x) = f(x)$$

we will let y_1 & y_2 be independent solutions of $Ly(x)=0$.

note: $y_1'' + P(x)y_1' + q(x)y_1 = f(x)$ &

$y_2'' + P(x)y_2' + q(x)y_2 = f(x)$.

The question is, can 2 functions of x be formed such that.

$$y_P = u_1(x)y_1 + u_2(x)y_2$$

Here we are making the assumption that replacing C_1 & C_2 with $u_1(x)$ & $u_2(x)$ will change y_c to y_P.

by differentiating y_P

$$y_P = u_1'(x)y_1 + u_1(x)y_1' + u_2'(x)y_2 + u_2(x)y_2'$$

we make the following assumption on u_1 & u_2 such that:

$$y_1 u_1' + y_2 u_2' = 0$$

this assumption is based on the fact that u_1 and u_2 under the conditions set for. by

$$Ly(x) = f(x)$$

therefore

$$y_p' = y_1' u_1 + y_2' u_2$$

then we differentiate again

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

CONTINUED...

$$\boxed{y_p = u_1 y_1 + u_2 y_2}$$

$$y_1 u_1' + y_2 u_2' = 0$$

$$\boxed{y_p' = y_1' u_1 + y_2' u_2}$$

then

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

then substituting y_p, y_p', and y_p'' in $Ly(x) = f(x)$

$$= Ly(x) = f(x)$$
$$= (y'' + py' + qy) = f(x)$$

$$= (u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2') + p(y_1' u_1 + y_2' u_2)$$
$$+ q(u_1 y_1 + u_2 y_2) = f(x)$$

rearranging we get.

$$= u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy)$$
$$+ u_1' y_1' + u_2' y_2' = f(x)$$

$$\therefore \quad u_1' y_1' + u_2' y_2' = f(x)$$
$$u_1' y_1 + u_2' y_2 = 0 \quad \Big\} \text{ assumed without proof.}$$

From 2 equations we can find 2 unknowns $\therefore$

we can solve down to

$$u_1' = \frac{y_2 f(x)}{y_1' y_2 - y_1 y_2'} = \frac{-y_2 f(x)}{y_2' y_1 - y_1' y_2} = \frac{-y_2 f(x)}{W(x)}.$$

$\therefore$

$$u_2' = \frac{+y_1 f(x)}{W(x)}$$

integrate to find u_1 & u_2

then the complete solution becomes,

$$y = \underbrace{C_1 y_1 + C_2 y_2}_{y_c} + \underbrace{u_1 y_1 + u_2 y_2}_{y_p.}$$

we are not going to add a constant of integration here...

EX,

$$4y'' + 36y = \operatorname{cosec} 3x.$$

$$y'' + 9y = \frac{\operatorname{cosec} 3x}{4}$$

find the auxilliary equation.

$$m^2 + 9 = 0 \qquad \therefore \quad m = \pm 3i$$

$$y_c = C_1 \cos 3x + C_2 \sin 3x$$

$$y_1 = \cos 3x \qquad y_2 = \sin 3x$$

we know that

$$W(x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 x + 3\sin^2 x = \underline{\underline{3}}$$

$$\therefore \quad u_1 = \int \frac{-y_2 f(x)}{W(x)}$$

$$= \int \frac{-\sin 3x \left(\frac{1}{4} \operatorname{cosec} 3x\right)}{3} dx$$

$$= \int -\frac{1}{12}\,dx = -\frac{1}{12}x$$

$$u_2 = \int \frac{y_1 f(x)}{\omega(x)} = \int \frac{(\cos 3x)(\frac{1}{4}\operatorname{cosec} 3x)}{3}$$

$$= \frac{1}{36}\ln|\sin 3x|$$

therefore the complete answer is.

$$y = C_1\cos 3x + C_2\sin 3x - \frac{1}{12}x\cos 3x$$

$$+\frac{1}{36}\ln|\sin 3x|\sin 3x$$

this can be rearranged. to.

$$y = \left(C_1 - \frac{1}{12}x\right)\cos 3x + \left(C_2 + \frac{1}{36}\ln|\sin 3x|\right)\sin 3x$$

here you can see that the constants of integration from integrating u_1' & u_2' are absorbed by C_1 & C_2.

EX. solve $y'' - y = \frac{1}{x}$

i.e.- find y_c & y_p (y_p by variation of parameters)

SOLUTION: find the aux eqn.

$$m^2 - 1 = \frac{1}{x} \quad \therefore \quad y_c = C_1 e^{x} + C_2 e^{-x}$$

find the wronskin

$$\omega(x) = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -(e^{x})(e^{-x}) - (e^{-x})(e^{x})$$

$$-1 - 1 = -2$$

$\therefore$

$$u_1 = \int -\frac{y_2 f(x)}{W(x)} = \int \frac{-e^{-x}\left(\frac{1}{x}\right)}{-2}\,dx$$

$$= \frac{1}{2}\int \frac{e^{-x}}{x}\,dx$$

$$u_2 = \int \frac{y_1 f(x)}{W(x)} = -\frac{1}{2}\int \frac{e^{x}}{x}\,dx$$

we will leave the integrals as is and our complete solution becomes.

$$y = C_1 e^{x} + C_2 e^{-x} + e^{x}\left[\frac{1}{2}\int \frac{e^{-x}}{x}\,dx\right] + e^{-x}\left[-\frac{1}{2}\int \frac{e^{x}}{x}\,dx\right]$$

OPERATOR METHODS

for the following y_p of non-homogeneous DE.

$$L = a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0$$

L is a differential opperator of the n^{th} order.

for $n = 2$ $\quad L = a_2 D^2 + a_1 D + a_0$

all the a's are constant, the differential opperators, because they are linear, can be added together by super position.

Let L_1 & L_2 be linear differential opperators, then

$$(L_1 + L_2)y = L_1 y + L_2 y$$

Ex.

$$((D^2 + D)y + (D-1)y) = (D^2 + 2D - 1)y$$

If $f(x)$ and L is an opperator then

$$f(x)\,L\,y(x) = f(x)\left[L\,y(x)\right]$$

If $f(x)$ is a constant then we say $f(x)=c$

$$[cL]\,y(x) = c\left[L\,y(x)\right]$$

Ex. $\left[(x^2-1)(D^2+2D)\right]y(x) = (x^2-1)\left[(D^2+2D)\,y(x)\right]$

MULTIPLICATION

$$(L_1 L_2)\,y(x) = L_1\underbrace{\left(L_2\,y(x)\right)}_{u(x)} = f(x)$$

$$L_1\,u(x) = f(x)$$

If L is a quadradic polynomial, $L = P(D)$, if we factorise $P(D)$ into $(D-a)(D-b)$, then $Ly = f(x)$ becomes:

$$\left[(D-a)(D-b)\right] y = f(x)$$

which is the same as

$$(D-a)\left[(D-b)y\right] = f(x)$$

Let $u(x) = (D-b)y$, then,

$$(D-a)\,u(x) = f(x)$$

which is a first order linear DE, (we can use Integrating factor)

$$I(x) = e^{-ax}$$

the solution is then

$$ue^{-ax} = \int f(x)\,dx + C$$

we may write for short

$$u(x) = (D-b)\,y(x) = \frac{1}{(D-a)} f(x)$$

or

$$u(x) = e^{ax}\int f(x)\,dx + Ce^{ax}$$

the particular solution

$$(D-b)\,y = e^{ax}\int e^{-ax} f(x)\,dx$$

we may write

$$y(x) = \frac{1}{(D-b)}\, g(x)$$

and

$$y(x) = e^{bx} \int e^{-bx} g(x)\,dx$$

and the find solution of the particular solution of $Ly = f(x)$, is y_P.

EX. find y_P.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 8\cos x$$

$$(D^2 - 2D)\,y = 8\cos x$$

$$(D)(D-2)\,y = 8\cos x$$

let $u = (D-2)y$, then

$$Du = 8\cos x$$

$$u = \int 8\cos x$$

$$u = 8\sin x + C$$

therefore

$$(D-2)\,y = 8\sin x + C$$

$$Dy - 2y = 8\sin x + C$$

$$I(x) = e^{-2x}$$

$$ye^{-2x} = \int 8e^{-2x}\sin x + C$$

$$y_P = e^{2x}\, 8 \int e^{-2x}\sin x$$

integrating by parts we get.

$$y_P = -\frac{8}{5}\left[2\sin x + \cos x\right] \Big\}\ \text{particular solution.}$$

Ex. Find y_P of

$$(D+1)^2 y = x^3 e^{-x}$$

$$(D+1) y_P = \frac{1}{D+1}\left[x^3 e^{-x}\right]$$

$$(D+1) y_P = e^{-x} \int e^{x} (x^3 e^{-x})\, dx$$

$$= e^{-x} \left(\left(\frac{x^4}{4}\right)\right)$$

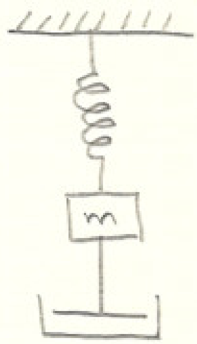
$$y_P = \frac{1}{D+1}\left[\frac{e^{-x} x^4}{4}\right]$$

$$= e^{-x} \int \frac{e^{x}}{4} x^4 e^{-x}\, dx$$

$$= \frac{x^5 e^{-x}}{20}$$

APPLICATION OF A 2^{nd} ORDER DE.

free motion, an example will be a mass spring assembly shown below



$$my'' + Cy' + ky = 0$$

Cy' = damping force

ky is spring force.

my'' is Inertial force.

If a external force $r(t)$ is acting on the mass m, then the DE is modified.

$$my'' + Cy' + ky = r(t).$$

where $y = y(t)$, $r(t)$ is known as the input or driving force.

One particular solution of $r(t) = F_0 \cos \omega t$ periodic function where ω is angular velocity, t is time.

SOLUTION

Solve for y_c ...

probably the best method to solve for y_p would be the method of undertermined coeffients

BACK TO OPERATOR METHOD FOR FINDING y_p.

L is quadradic and can be factorised as

$$L = (D-a)(D-b)$$

and our non-homogeneous DE

$$Ly = f(x)$$
$$(D-a)(D-b)y = f(x)$$

let $(D-b)y = u(x)$

$$\therefore \quad (D-a)u(x) = f(x)$$
$$I(x) = e^{\int -a\,dx} = e^{-ax}$$

$$u(x)e^{-ax} = \int e^{-ax} f(x)\,dx$$

$$u(x) = \underbrace{e^{ax}\int e^{-ax} f(x)\,dv}_{g(x)} = \frac{1}{D-a} f(x)$$

$$u = (D-b)y$$
$$(D-b)y = g(x)$$

$$y_p = \frac{1}{D-b} g(x)$$

$$= e^{bx} \int e^{-bx} g(x)\, dx$$

the constant C is made zero b/c we are looking for a particular solution.

OPEATOR METHOD

EX solve for y_p.

$$(D^2 - D - 6)y = e^{5x}$$

SOL:

$$y = \frac{e^{5x}}{D^2 - D - 6}$$

replace D by 5.

$$y = \frac{e^{5x}}{(5)^2 - (5) - 6}$$

EX

$$(D^2 + 2D + 5)y = 4e^{-2x} + 10$$

$$y = \underbrace{\frac{4e^{-2x}}{D^2 + 2D + 5}}_{\text{replace D's with } (-2)} + \underbrace{\frac{10}{D^2 + 2D + 5}}_{\text{replace D's with zero.}}$$

LAPLACE TRANSFORM.

Definition, let $f(t)$ be real, valued function in $0 \le t \le \infty$

the laplace transform, if it exists ($\mathcal{L}$)

$$\mathcal{L} f(t) = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

similarly.

$$\mathcal{L} y(t) = Y(s).$$

the Laplace transform is regarded as an operator.

LAPLACE TRANSFORM OF SIMPLE EQU.

$f(t) = 1$

$$\mathcal{L} f(t) = \int_0^\infty e^{-st}(1)\,dt. = -\frac{e^{-st}}{s}\Bigg|_0^\infty = \frac{1}{s}$$

In general $f(t) = C$; then $F(s) = \frac{C}{s}$

$f(t) = e^{at}$

$$\mathcal{L} f(t) = \int_0^\infty e^{-st}\cdot e^{at}\,dt = \int_0^\infty e^{-t(s-a)}\,dt$$

$$= \frac{-1}{s-a}\left[e^{-t(s-a)}\right]_0^\infty = \frac{1}{s-a}$$

IN general $f(t) = e^{at}$; $F(s) = \frac{1}{s-a}$

CONSIDER THE FOLLOWING.

$f(t) = \sin kx$

$f(t) = \cos kx$

$f(t) = e^{ikt}$

$$\mathcal{L} f(t) = \int_0^\infty e^{ikt} e^{-st}\,dt = \int_0^\infty e^{-t(s-ik)}\,dt$$

$$= \frac{-1}{s - ik} \cdot \frac{s + ik}{s + ik}$$

$$= \frac{s + ik}{s^2 + k^2} = \frac{s}{s^2 + k^2} + i \frac{k}{s^2 + k^2}$$

which is the same as each trig term from tables.

$$f(t) = \sinh(kt)$$

$$\mathcal{L} f(t) = \frac{1}{2} \int_0^\infty (e^{kt} - e^{-kt}) e^{-st} \, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t(s-k)} - e^{-t(s+k)} \, dt$$

$$= \frac{1}{2} \left[\frac{-e^{-t(s-k)}}{s-k} + \frac{e^{-t(s+k)}}{s+k} \right]_0^\infty$$

$$= \frac{k}{s^2 - k^2}$$

similarly $f(t) = \cosh t$; then $F(s) = \frac{s}{s^2 - k^2}$

$$f(t) = t$$

$$\mathcal{L} f(t) = \int_0^\infty t e^{-st}$$

integrate by parts.

$$F(s) = -\frac{1}{s^2}$$

IN general $f(t) = t^n$; $F(s) = \frac{(n-1)!}{s^n}$

LAPLACE TRANSFORM OF DERIVATIVES.

functions are continuous.

$$\mathcal{L} f'(t) = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv}$$

$$du = -se^{-st}\,dt \qquad v = f(t).$$

$$F(s) = \left[\left[e^{-st} f(t)\right]_0^\infty + s\int f(t)e^{-st}\,dt \right]$$

$$= \left[e^{-\infty} f(\infty) - e^0 f(0) \right] + sF(s)$$

$$= sF(s) - f(0)$$

$$\mathcal{L} f^n(t) = \int_0^\infty e^{-st} f^n(t)\,dt$$

$$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0)$$

$$- f^{n-1}(0).$$

INVERSE LAPLACE TRANSFORMS.

$$\mathcal{L} f(t) = F(s)$$

the opperator ($\mathcal{L}$) changes a function from the t domain, to the s domain, which is known as the complex frequency plane.

the inverse $\mathcal{L}^{-1}$, changes a function from s domain, to the t domain.

$$\mathcal{L}^{-1} F(s) = f(t)$$

There are advanced methods to obtain f(t) from F(s). In this course we will be using tables.

EX

$$\mathcal{L}^{-1}\left(\frac{c}{s}\right) = c$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$$

IN GENERAL

$$\mathcal{L}^{-1}\left\{ c_1 F_1(s) + c_2 F_2(s) \right\}$$

where c_1 & c_2 are constent, the above can be written as.

$$c_1 \mathcal{L}^{-1}\left\{F_1(s)\right\} + c_2 \mathcal{L}^{-1}\left\{F_2(s)\right\}$$

FIRST TRANSLATION THEROEM

$$\mathcal{L}\, e^{at} f(t) = F(s-a)$$

$$e^{at} f(t) = \mathcal{L}^{-1}\left\{F(s-a)\right\}$$

EX

$= \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2 - 9}\right\}$ using the first translarstion therom.

$= e^{-t} \sin 3t$

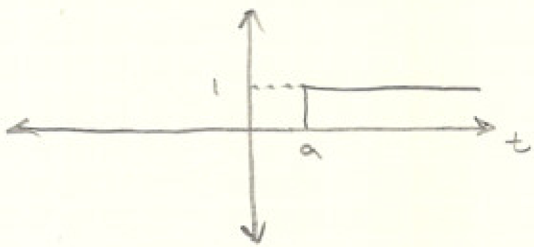
EX. find the inverse of

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+6s+11}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s+3 \; -3}{(s+3)^2 + 2}\right]$$

$$e^{-3t}\cos\sqrt{2}t - \frac{3}{\sqrt{2}}e^{-3t}\sin\sqrt{2}t$$

UNIT STEP FUNCTION.



$u(t-a)$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

often in engineering we encounter functions that are either off or on.

2ND TRANSFER FUNCTION.

$$\mathcal{L}[f(t-a)\,u(t-a)] = e^{-as}\mathcal{L}\,f(t)$$

Ex Evaluate.

$$\mathcal{L}\,u(t-5) = e^{-5s}\mathcal{L}(1) = \frac{e^{-5s}}{s}$$

Ex Evaluate.

$$\mathcal{L}[\sin(t-2\pi)\,u(t-2\pi)] = e^{-2\pi s}\mathcal{L}(\sin t)$$

Ex Evaluate Inverse.

$$\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)\,u(t-a)$$

Ex Evaluate Inverse

$$= \mathcal{L}^{-1}\left[\frac{e^{-\pi s/2}}{s^2+9}\right] \qquad \text{where } t = t - \pi/2$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] = \frac{1}{3}\sin 3\left(t-\frac{\pi}{2}\right)u\left(t-\frac{\pi}{2}\right)$$

EX. Solve the following problem using L.T.

$$y' - 2y = 4$$

$$\mathcal{L}[y'] - 2\mathcal{L}[y] = 4\mathcal{L}[(1)]$$

$$sY(s) - y(0) - 2Y(s) = \frac{4}{s}$$

$$(s-2)Y(s) = \frac{4}{s} \qquad Y(s) = \frac{4}{s(s-2)}$$

now we use partial fractions to solve inverse.

$$\frac{4}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$4 = A(s-2) + Bs$$

$$\text{let } s=0, \quad A=-2$$

$$s=2, \quad B=2$$

$$Y(s) = \frac{-2}{s} + \frac{2}{s-2}$$

$$= -2 + 2e^{2t}$$

LAPLACE TRANSFORM SOLUTITION OF IVP.

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 = g(x)$$

and all the Initial values are given.

$$y(0) = y_0, \quad y'(0) = y_0'; \dots \quad y^n(0) = y_0^{(n)}$$

then we can Laplace each term, rendering

$$a_n \mathcal{L}\left[\frac{d^n y}{dt^n}\right] + \dots + a_1 \mathcal{L}\left[\frac{dy}{dt}\right] + a_0 \mathcal{L}[y] = \mathcal{L}[g(x)]$$

if we ~~set~~ $n=2$ we can get.

$$a_2\left[s^2 Y(s) - s y_0 - y_0'\right] + a_1\left[sY(s) - y_0\right] + a\left[Y(s)\right] = G(s)$$

then rearrange for $Y(s)$, fiddle and solve for inverse L.T.